Hence, as x increases from 0 to ∞, it follows that $V_{1/3}(x)/U_{1/3}(x)$ steadily increases from $-\cot \frac{5}{12}\pi$ to 0, the limit of $V_{1/3}(x)/U_{1/3}(x)$ when $x \rightarrow 0$ being derived from its expression in terms of $J_{\pm 1/3}(x)$.

Now the approximate expression for $J_n\,(n \sec \beta)$ may be written in the form

$$J_n\,(n \sec \beta) \sim 3^{-\frac{1}{2}} \tan \beta \cos \{n\,(\tan \beta - \beta) - \tfrac{1}{4}\pi\}\, U_{1/3}\,(\tfrac{1}{3} n \tan^3 \beta)$$
$$- 3^{-\frac{1}{2}} \tan \beta \sin \{n\,(\tan \beta - \beta) - \tfrac{1}{4}\pi\}\, V_{1/3}\,(\tfrac{1}{3} n \tan^3 \beta).$$

Hence, when $\beta \leq \frac{1}{4}\pi$, the zeros of $J_n\,(n \sec \beta)$ are given approximately by the equation

$$\cot \{n\,(\tan \beta - \beta) - \tfrac{1}{4}\pi\} = V_{1/3}\,(\tfrac{1}{3} n \tan^3 \beta)/U_{1/3}\,(\tfrac{1}{3} n \tan^3 \beta).$$

If we draw the graphs of the functions

$$V_{1/3}\,(\tfrac{1}{3} n \tan^3 \beta)/U_{1/3}\,(\tfrac{1}{3} n \tan^3 \beta), \qquad \cot \{n\,(\tan \beta - \beta) - \tfrac{1}{4}\pi\},$$

we see that, near the intersection of the graphs the former has a positive slope and the latter a negative slope, and a good estimate of the value of $\frac{1}{3} n \tan^3 \beta$ at the intersection can be obtained by taking values of β on each side of the intersection and then using the principle of proportional parts.

The Electrostatic Problem of a Conducting Sphere in a Spherical Cavity.

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The mathematical problems which arise in connection with spherical electrodes have often been discussed. A *résumé* of them is given in Chap. VIII, Vol. I, of the second edition of the author's treatise on 'Alternating Current Theory.' The problem, however, of a sphere in a spherical cavity has barely been considered. Sir W. Thomson* (Lord Kelvin) gives numerical results which he says were calculated for application to the theory of conducting spheres "in the interior of a hollow insulated and electrified conductor." As a matter of fact, however, the numerical results given apply only to two equal conducting spheres external to one another.

Solutions for the capacity between two spheres have been given for the case when the centre of the sphere is close to the centre of the cavity. These solutions are either incorrect,* or approximately correct† only in special cases. It will be useful, therefore, to show how the complete numerical solution can be readily found in all cases.

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* 'Reprint,' p. 96.

The solution is required in connection with the following problems, which are of importance in physical and engineering science: (1) The calculation of the capacity of a spherical condenser when the centre of the inner sphere does not coincide with the centre of the cavity; (2) the calculation of the electrostatic force acting on the inner sphere; and (3) the calculation of the maximum value of the electric stress on the dielectric. The first problem arises when determining the ratio of the electrostatic to the electromagnetic unit of electricity by means of a spherical condenser. The solution is also of value when constructing spherical condensers of variable capacity. The solution of the second problem is of value, as it enables us to make an absolute electrometer' by suspending one sphere inside another. The attracting force for a given distance between them is greater in certain cases than when the spheres are external to one another, and errors due to the presence of conducting or insulating materials in the neighbourhood are obviated. The final problem—the calculation of the maximum value of the potential gradient between the two spheres—arises when determining the electric strengths of insulating materials by using them as the dielectric of a condenser. The electric pressure between the electrodes is increased at a definite rate, and the value of the disruptive voltage is noted. The determination of the electric strengths of dielectrics is a problem of great urgency. Methods of finding these values are at present being considered and tested experimentally by Panels of the Research Committee of the Institution of Electrical Engineers.

1. *The Capacity between the Two Spheres.*

Let q_1 and q_2 be the total charges on the inner and outer conductors respectively, and let v_1 and v_2 be their potentials. If k_{11} and k_{22} be their self-capacity coefficients, and k_{12} be the mutual capacity coefficient, we may write

$$q_1 = k_{11}v_1 + k_{12}v_2, \tag{1}$$

and

$$q_2 = k_{22}v_2 + k_{12}v_1. \tag{2}$$

Since q_1 must vanish when $v_1 = v_2$, we have $k_{11} = -k_{12}$.

It is obvious that $-q_1$ is the charge on the inner side of the outer conductor, and therefore q_1+q_2 is the charge on the outer surface. Hence, if K be the capacity of the outside surface, we have

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* J. H. Jeans, 'Electricity and Magnetism,' 2nd ed. ex. 47, p. 292; L. Cohen, Alternating Current Problems,' p. 90.

† E. B. Rosa and N. E. Dorsey, "Bulletin of the Bureau of Standards," "The Determination of the Ratio of the Electric Units," vol. 3, p. 477.

$$q_1+q_2 = \mathrm{K}v_2.$$

But from (1) and (2)

$$q_1+q_2 = (k_{22}-k_{11})\,v_2,$$

and hence

$$k_{22} = \mathrm{K}+k_{11}, \tag{3}$$

and

$$q_1 = k_{11}(v_1-v_2).$$

In the problem we are considering the value of K is immaterial, and k_{11} is the capacity C of the condenser formed by the inner and outer conductors.

Let a and b be the radii of the sphere and of the cavity respectively, and let c be the distance between their centres. By Kelvin's method of images we get

$$\mathrm{C} = a+\frac{a^2b}{d^2}+\frac{a^3b^2}{d^4-a^2c^2}+\frac{a^4b^3}{d^2(d^4-2a^2c^2)-a^4c^2}$$

$$+\frac{a^5b^4}{d^4(d^4-3a^2c^2)-2a^4c^2d^2+a^4c^2(c^2-a^2)}+\mathrm{R}_5, \tag{4}$$

where $d^2 = b^2-c^2$, and R_5 denotes the remainder of the series after the first five terms. If we write $-b$ for b in (4), we get the well known formula for k_{11} for two external spheres. Except when both a and c are very small compared with b, (4) is of little use for numerical computation.

If we introduce r, the radius of the sphere which cuts the two spheres orthogonally, and two new variables α and β defined by the equations,

$$\sinh\alpha = r/a, \qquad \sinh\beta = r/b, \tag{A}$$

into the formula it simplifies considerably.

It is also convenient to denote $\alpha-\beta$ by ω. It is easy to show that

$$c = (b^2+r^2)^{\frac{1}{2}}-(a^2+r^2)^{\frac{1}{2}}, \tag{B}$$

$$\left.\begin{aligned}\cosh\alpha &= (b^2-a^2-c^2)/2ac\\ \cosh\beta &= (b^2+c^2-a^2)/2bc\\ \text{and}\qquad \cosh\omega &= (a^2+b^2-c^2)/2ab\end{aligned}\right\}. \tag{C}$$

Writing $2s$ for $a+b+c$ we get

$$ab\sinh\omega = cr = 2\{s(s-a)(b-s)(s-c)\}^{\frac{1}{2}}, \tag{D}$$

$$\left.\begin{aligned}&\sinh(\omega/2) = \{(s-a)(b-s)/ab\}^{\frac{1}{2}}; \qquad \cosh(\omega/2) = \{s(s-c)/ab\}^{\frac{1}{2}};\\ &\text{and}\\ &\sinh(\alpha/2) = \{s(b-s)/ac\}^{\frac{1}{2}}; \qquad \cosh(\alpha/2) = \{(s-a)(s-c)/ac\}^{\frac{1}{2}}\end{aligned}\right\}. \tag{E}$$

We also have

$$\omega = 2\log_e\{\cosh(\omega/2)+\sinh(\omega/2)\}.$$

These formulæ, which are adapted for logarithmic computation, enable us to calculate the values of α, β, ω, and r for any given values of a, b, and c easily and rapidly.

In terms of r, α and ω, (4) becomes

$$C = r \sum_{n=0}^{n=\infty} \frac{1}{\sinh(\alpha+n\omega)}. \qquad (5)$$

It will be seen that C, which equals k_{11}, is given by Maxwell's formula, but it must be carefully noticed that $\omega = \alpha - \beta$, and can therefore never be greater than α. In the case of two external spheres, the author (*loc. cit. ante*) has shown how the series in (5) can be computed in all cases. But for external spheres $\omega = \alpha + \beta$, and hence ω cannot be less than α. In the problem under consideration ω cannot be greater than α, and so some of the formulæ we have previously given can only be used for this problem in special cases. We have to find, therefore, suitable formulæ for the other cases. It is easy to prove from (C) that as c increases from zero to $b-a$, α diminishes from infinity to zero, and ω diminishes from $\log(b/a)$ to zero. In this case also ω/α increases from zero to c/b.

When α is large C can be found from (4) and (5). For instance, if $\alpha+n\omega$ is not less than 5, we can write $\cosh(\alpha+n\omega) = \sinh(\alpha+n\omega)$, the error being less than 1 in 10,000. Hence, to this accuracy at least,

$$R_n = r/\sinh(\alpha+n\omega) \,.\, e^{\omega}/(e^{\omega}-1), \qquad (6)$$

where

$$e^{\omega}/(e^{\omega}-1) = \tfrac{1}{2} + \tfrac{1}{2}\{[(b+a)^2-c^2)/[(b-a)^2-c^2]\}^{\frac{1}{2}}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\{s(s-c)/(s-a)(b-s)\}^{\frac{1}{2}}.$$

When α is not less than 5, and b^2 is therefore not less than a^2+c^2+10ac, we have

$$C = \frac{r}{\sinh\alpha} \cdot \frac{e^{\omega}}{e^{\omega}-1} = \frac{a}{2} + \frac{a}{2}\left\{\frac{s(s-c)}{(s-a)(b-s)}\right\}^{\frac{1}{2}}.$$

Finally, when c is zero,

$$C = ab/(b-a).$$

When α and ω are small, a series very suitable for computing purposes can be found as follows. Expanding the terms in (5) in powers of $e^{-\alpha}$ and $e^{-\omega}$, and summing the series in a different order, we get

$$R_n = a(1-e^{-2\alpha})e^{-n\omega}\{1/(1-e^{-\omega}) + e^{-2(\alpha+n\omega)}/(1-e^{-3\omega})$$
$$+ e^{-4(\alpha+n\omega)}/(1-e^{-5\omega}) + \ldots\}. \qquad (7)$$

Except when α and $n\omega$ are very minute, this series converges with most satisfactory rapidity. It can be used in nearly every practical case.

Another useful expression which enables us to compute C rapidly can be obtained as follows. Writing D for $\partial/\partial\alpha$, we get by Taylor's Theorem

$$\begin{aligned} f(\alpha)+f(\alpha+\omega)+f(\alpha+2\omega)+\ldots &= \{1+e^{\omega \mathrm{D}}+e^{2\omega \mathrm{D}}+\ldots\}f(\alpha) \\ &= \frac{1}{1-e^{\omega \mathrm{D}}}f(\alpha) \\ &= \mathrm{A}-\frac{1}{\omega}\int f(\alpha)\,\partial\alpha+\tfrac{1}{2}f(\alpha)-\frac{\mathrm{B}_1\omega}{2\,!}f'(\alpha) \\ &\quad +\frac{\mathrm{B}_3\omega^3}{4\,!}f'''(\alpha)-\ldots, \qquad \text{(F)} \end{aligned}$$

where B_1, B_3,... are Bernoulli's numbers, and A is a constant which has to be determined in any particular case. The limitations of this theorem can be stated by aid of a theorem given in Whittaker and Watson's 'Modern Analysis,' § 7·21. For our purpose it is sufficient to remember that the series on the right-hand side is semi-convergent, but in any case in which it is applicable the value of the last term included in the computation enables us to see the accuracy obtained.

Applying theorem (F) to formula (5), we get

$$\begin{aligned} \mathrm{C} = \frac{r}{\omega}\Big\{&\tfrac{1}{2}\log\frac{\cosh\alpha+1}{\cosh\alpha-1}+\frac{\omega}{2\sinh\alpha}+\frac{\omega^2}{12}\cdot\frac{\cosh\alpha}{\sinh^2\alpha} \\ &-\frac{\omega^4}{720}\cdot\frac{\cosh\alpha}{\sinh^2\alpha}\cdot\frac{\sinh^2\alpha+6}{\sinh^2\alpha} \\ &+\frac{\omega^6}{30240}\cdot\frac{\cosh\alpha}{\sinh^2\alpha}\cdot\frac{\sinh^4\alpha+60\sinh^2\alpha+120}{\sinh^4\alpha}-\ldots\Big\}, \qquad (8) \end{aligned}$$

the constant A being equal to zero. That A vanishes can easily be seen by making α very great in (8).

In many cases the simpler formula

$$\begin{aligned} \mathrm{C} = \frac{r}{\omega}\Big\{&\frac{1}{\cosh\alpha}+\frac{1}{3\cosh^3\alpha}+\frac{1}{5\cosh^5\alpha}+\ldots\Big\} \\ &+\frac{a}{2}+\frac{\omega a}{12}\cdot\frac{\cosh\alpha}{\sinh\alpha}-\left(\frac{\omega a}{12}\cdot\frac{\cosh\alpha}{\sinh\alpha}\right)\frac{\omega^2}{60}\cdot\frac{\sinh^2\alpha+6}{\sinh^2\alpha}+\ldots \qquad (9) \end{aligned}$$

enables us to find C with high accuracy.

If we put $c = 0$, we get from (8)

$$\mathrm{C} = a\left[\frac{1}{\omega}+\frac{1}{2}+\frac{\mathrm{B}_1}{2\,!}\,\omega-\frac{\mathrm{B}_3}{4\,!}\,\omega^3+\ldots\right] = a\,e^{\omega}/(e^{\omega}-1) = ab/(b-a).$$

When α is small the following formula readily deducible from (8) may be used

$$\begin{aligned} \mathrm{C} = \frac{r}{\omega}\Big\{&-\psi\left(\frac{\alpha}{\omega}\right)+\log\frac{2}{\omega}+\frac{\omega^2}{72}+\frac{7\,\omega^4}{43200}+\frac{31\,\omega^6}{3810240}+\ldots \\ &+\frac{\alpha\beta}{12}-\frac{7\alpha^2\beta^2}{1440}+\frac{31\alpha^3\beta^3}{90720}-\frac{31\alpha^2\beta^2\omega^2}{181440}+\ldots\Big\}, \qquad (10) \end{aligned}$$

where $\psi(x)$ is the logarithmic derivate of the gamma function.

Since $C = k_{11}$, and for an infinitely thin shell $k_{22} = C+b$, we get by symmetry,

$$C+b = \frac{r}{\omega}\left\{-\psi\left(\frac{\beta}{\omega}\right)+\log\frac{2}{\omega}+\frac{\omega^2}{72}+\frac{7\omega^4}{43200}+\frac{31\omega^6}{3810240}+\dots\right.$$

$$\left.+\frac{\beta(\beta-\omega)}{12}-\frac{7\beta^2(\beta-\omega)^2}{1440}+\frac{31\beta^3(\beta-\omega)^3}{90720}-\frac{31\beta^2\omega^2(\beta-\omega)^2}{181440}-\dots\right\}. \quad (11)$$

If we subtract (10) from (11), and notice that $\alpha = \beta+\omega$, we get, as we ought,

$$b = r\left\{\frac{1}{\beta}-\frac{\beta}{6}+\frac{7\beta^3}{360}-\frac{31\beta^5}{15120}+\dots\right\} = \frac{r}{\sinh\beta}.$$

For computing purposes when α and ω are small, either equation (10) or (11) may be used. In the particular case when $\beta = \omega$, and therefore $c = a$, we get

$$C = -b+\frac{r}{\omega}\left\{\gamma+\log\frac{2}{\omega}+\frac{\omega^2}{72}+\frac{7\omega^4}{43200}+\frac{31\omega^6}{3810240}+\dots\right\}, \quad (12)$$

where $\gamma = 0{\cdot}5772157\dots$ (Euler's constant).

The series (10) and (11) can also be readily deduced from the general theorem (15) given in a preceding paper.* For the function $F(x)$ previously used we have written $-\psi(1/x)-\log x$. Tables of the values of $\psi(x)$ are published.†

When the distance x between the spheres is small compared with $b-a$, the maximum possible distance between their centres, we get, on substituting $b-a-x$ for c in formulæ (C) and (D), and expanding in powers of $x/(b-a)$, the following approximate equations :—

$$r^2 = \frac{2abx}{b-a}\left\{1+\frac{a^2+ab+b^2}{2ab}\cdot\frac{x}{b-a}+\frac{(a+b)^2}{2ab}\cdot\frac{x^2}{(b-a)^2}\right\}, \quad (13)$$

$$\frac{\alpha}{\omega} = \frac{b}{b-a}+\frac{b+a}{3(b-a)}\cdot\frac{x}{b-a}+\frac{(b+a)\{25ab-2(a^2+b^2)\}}{90ab(b-a)}\cdot\frac{x^2}{(b-a)^2}. \quad (14)$$

and writing $k = 1+(b-a)^2/3ab$, we get

$$\frac{r}{\omega} = \frac{ab}{b-a}\left\{1+k\frac{x}{b-a}+\frac{3+9k-2k^2}{10}\cdot\frac{x^2}{(b-a)^2}\right\}. \quad (15)$$

Neglecting the sixth and higher powers of small quantities in (10) we get

$$C = \frac{r}{\omega}\left\{\tfrac{1}{2}\log\frac{2ab}{(b-a)x}-\psi\left(\frac{\alpha}{\omega}\right)+\tfrac{1}{3}(k+\tfrac{1}{4})\frac{x}{b-a}\right.$$

$$\left.+\frac{1}{3600}(331+948k-624k^2)\frac{x^2}{(b-a)^2}\right\}. \quad (16)$$

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* 'Roy. Soc. Proc.,' A, vol. 82, p. 527 (1909).

† *E.g.*, in Russell's 'Alternating Currents,' 2nd ed., vol. 1, p. 241.

The values of r/ω and α/ω can be found from (14) and (15), and there is no difficulty in calculating the value of $\psi(x)$ in every case by means of the formulæ

$$\psi(x) = \log x - 1/2x - \mathrm{B}_1/2x^2 + \mathrm{B}_3/4x^4 - \mathrm{B}_5/6x^6 + \ldots \tag{G}$$

and

$$\psi(x) = \psi(1+x) - 1/x. \tag{H}$$

When x is so small that $x/(b-a)$ can be neglected compared with unity, we get

$$\mathrm{C} = \frac{ab}{b-a}\left\{\tfrac{1}{2}\log\frac{2ab}{(b-a)x} - \psi\left(\frac{b}{b-a}\right)\right\}. \tag{17}$$

As an illustration of the application of these formulæ, let us first calculate the capacity of the spherical condenser B used by Rosa and Dorsey in their experiments for the determination of v. The dimensions of this condenser are given on p. 459 of their classical paper (*loc. cit. ante*). At a certain temperature they found that $b = 12{\cdot}67158$ and $a = 8{\cdot}87391$ cm. respectively. When the inner sphere is central we should therefore have

$$\mathrm{C} = 29{\cdot}60933\ldots = \mathrm{C}_0 \text{ (say)}.$$

They give measurements of the capacity of the condenser when $c/(b-a)$ is small, and they also give the following formula,

$$\mathrm{C} = \mathrm{C}_0\left\{1 + \tfrac{1}{3}\cdot\frac{c^2}{(b-a)^2} + \tfrac{1}{5}\cdot\frac{c^4}{(b-a)^4}\right\}. \tag{18}$$

for calculating C in this case.

Taking the case when $c = 0{\cdot}3$ cm., and using (18), we get

$$\mathrm{C} = \mathrm{C}_0\{1 + 0{\cdot}002080 + 0{\cdot}000008\} = 29{\cdot}67115.$$

We shall apply some of the formulæ given above to check this result. Putting $n = 0$ in (7) we get

$$\mathrm{C} = 2r\{e^{-\alpha}/(1-e^{-\omega}) + e^{-3\alpha}/(1-e^{-3\omega}) + e^{-5\alpha}/(1-e^{-5\omega}) + \ldots\}. \tag{19}$$

We easily find from (D) and (E) that

$$\log r = 2{\cdot}1333210, \quad \log e^{-\alpha} = \bar{2}{\cdot}5133018, \quad \text{and } \log e^{-\omega} = 1{\cdot}8457628.$$

Hence we get

$$\mathrm{C} = 2r\{0{\cdot}1090776 + 0{\cdot}0000529 + 0{\cdot}0000000 + \ldots\} = 29{\cdot}66861.$$

If we put $n = 2$ in (7), we get

$$\mathrm{C} = a + a^2b/(b^2-c^2) + 2r\{e^{-(\alpha+2\omega)}/(1-e^{-\omega}) + e^{-3(\alpha+2_e\omega)}/(1-e^{-3\omega}) + \ldots\}. \tag{20}$$

Hence, substituting the numerical values, we find that

$$\mathrm{C} = 8{\cdot}87391 + 6{\cdot}21788 + 2r\{0{\cdot}05361190 + 0{\cdot}0000063 + \ldots\} = 29{\cdot}66862.$$

Similarly, making $n = 3$ and substituting the numerical values in the resulting formula, we get the same answer. To get the same accuracy by Maxwell's series formula (5) we would have to find the value of about 40 terms of the series.

If we use formula (8) we get

$$C = 24{\cdot}969052 + 4{\cdot}436955 + 0{\cdot}263186 - 0{\cdot}000567 + 0{\cdot}000002 - \ldots$$
$$= 29{\cdot}66862.$$

It will be seen that the value computed by (18) is too great by about 8·5 in 100,000. From fig. 11 in the authors' paper the measured value seems to be about $C_0(1{\cdot}0023)$, giving an error, therefore, of about 3 parts in 10,000.

The true formula for C in a series of expanding powers of $c^2/(b-a)^2$ can be deduced without much difficulty from (19). We find that

$$C = C_0\{1 + A_2 c^2/(b-a)^2 + A_2A_4 c^4/(b-a)^4 + \ldots\}, \qquad (21)$$

where $$A_2 = ab/(a^2+ab+b^2),$$

and $$A_4 = \frac{ab(a^2+ab+b^2)^2 + (a^2+b^2)(b^2-a^2)^2}{(a^2+ab+b^2)\{(a^2+b^2)^2 + ab(a^2-ab+b^2)\}}.$$

This is a useful formula when $c^2/(b-a)^2$ is small.

For the values of a, b and c given above we get, by (21),

$$C = C_0\{1 + 0{\cdot}0019948 + 0{\cdot}0000077\} = 29{\cdot}668623.$$

When b and a are nearly equal to one another formula (21) is practically identical with (18). The formula, however, can only be used when we are neglecting the sixth and higher powers of $c/(b-a)$, and so c must be small compared with $b-a$.

As a further numerical example let us suppose that $b = 10$, $a = 7$, and $c = 1$. In this case $c/(b-a)$ is $\frac{1}{3}$ and we should expect that the value of C computed by (21) would be too small. In this case $\cosh\omega = 37/35$, $\sinh\omega = 12/35$, $\cosh\alpha = 25/7$ and $\sinh\alpha = 24/7$. We thus easily find by (9) that

$$C = 20{\cdot}51990 + 3{\cdot}5 + 0{\cdot}20445 - 0{\cdot}00058 + \ldots = 24{\cdot}22377.$$

By (21) we get $C = 24{\cdot}219$; the value is, therefore, too small by about 1 part in 5000.

Let us now suppose that $b = 109$, $a = 98$ and $c = 10{\cdot}8$. In this case $\alpha = 0{\cdot}2025243$, $\beta = 0{\cdot}1823216$, and consequently $\omega = 0{\cdot}0202027$. Maxwell's series would, therefore, converge so slowly that the labour involved in computing C by it would be prohibitive. Using (10) we get

$$C = r/\omega \,.\, \{-2{\cdot}2543372 + 4{\cdot}5950858 + 0{\cdot}0000057 + 0{\cdot}0000000 + \ldots$$
$$+ 0{\cdot}0030770 - 0{\cdot}0000066 + \ldots\} = 2318{\cdot}373.$$

Similarly by (11) we get

$$C+109 = r/\omega \,.\, \{-2{\cdot}1435290+4{\cdot}5950858+0{\cdot}0000057+0{\cdot}0000000+\dots$$
$$+0{\cdot}0024631-0{\cdot}00000425+\dots\},$$

and hence $$C = 2318{\cdot}373.$$

As a further numerical example let us suppose that $b = 5$, $a = 2$ and $c = 2$. In this case, $\cosh n\omega = 2^{n-1}+1/2^{n+1}$, and $\sinh n\omega = 2^{n-1}-1/2^{n+1}$. Hence $e^{\omega} = 2$ and $\omega = \log_e 2 = 0{\cdot}6931472$. We also have $r = 3{\cdot}75$. Substituting in (12) we get $C = 3{\cdot}891981$. In this case (10) gives the same answer with very little additional labour and (7) is even more suitable. The error introduced by using (21), however, is about 2 per cent., $c/(b-a)$ being $\frac{2}{3}$.

As a final example, let us suppose that $b = 201$, $a = 100$, and $c = 100$. We have $x = 1$, and hence substituting in (14) and (15) we get $r/\omega = 201{\cdot}33$ and $\alpha/\omega = 2{\cdot}0000$. Hence by (16)

$$C = 201{\cdot}33 \times 2{\cdot}5752 = 518{\cdot}465.$$

The true value of C found by (12) is 518·468. The agreement is therefore quite satisfactory. When the spheres are very close together, (17) gives accurate results. If we write $-b$ for b in (17) we get the value of k_{11} for two external spheres of radii a and b respectively at a microscopic distance x apart.*

2. *The Electrostatic Force on the Inner Sphere.*

Let W denote the electrostatic energy stored up between the two spheres, and let v_1 and v_2 be their potentials. If C be the capacity between them, we have

$$W = \tfrac{1}{2} C (v_1 - v_2)^2,$$

and thus, if F be the attractive force when the potential difference is maintained constant, we have

$$F = \tfrac{1}{2} (v_1 - v_2)^2 \, \partial C/\partial c.$$

Let us first suppose that $c/(b-a)$ is small compared with unity. In this case we get, by (21),

$$F = \frac{a^2 b^2 c \,(v_1 - v_2)^2}{(b^3 - a^3)(b-a)^2} \left\{ 1 + 2A_4 \frac{c^2}{(b-a)^2} \right\} \dots \qquad (22)$$

approximately. Hence, for small values of $c^2/(b-a)^2$, the attractive force varies as the distance between the centre of the sphere and the centre of the cavity.

It is easy to see, from the definitions of α, β, ω, and r, that

$$\partial\alpha/\partial c = -\sinh\alpha \cosh\beta / r \sinh\omega\,; \qquad \partial\beta/\partial c = -\sinh\beta \cosh\alpha / r \sinh\omega\,;$$
$$\partial\omega/\partial c = -1/r\,; \qquad \text{and} \qquad \partial r/\partial c = -\cosh\alpha \cosh\beta / \sinh\omega.$$

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* Russell, 'Alternating Currents,' 2nd. ed., vol. 1, p. 241.

We thus get by (5)

$$\frac{2\mathrm{F}}{(v_1-v_2)^2} = \sum_1^\infty n\frac{\cosh(\alpha+n\omega)}{\sinh^2(\alpha+n\omega)} - \frac{\cosh\alpha\cosh\beta}{\sinh\omega}\sum_0^\infty\frac{1}{\sinh(\alpha+n\omega)}$$
$$+\frac{\sinh\alpha\cosh\beta}{\sinh\omega}\sum_0^\infty\frac{\cosh(\alpha+n\omega)}{\sinh^2(\alpha+n\omega)}\dots \qquad (23)$$

When ω is small, the summation of the three series given in this formula is laborious. It is therefore better to convert them into semi-converging series by (F). We thus readily find that

$$\frac{2\mathrm{F}}{(v_1-v_2)} = \frac{1}{2\omega^2}\log\frac{\cosh\alpha+1}{\cosh\alpha-1} + \frac{\cosh\alpha}{12\sinh^2\alpha} - \frac{\omega^2}{240}\cdot\frac{\cosh\alpha(\cosh^2\alpha+5)}{\sinh^4\alpha} + \dots$$
$$-\frac{\cosh\beta}{\sinh\omega}\left\{\frac{\cosh\alpha}{2\omega}\log\frac{\cosh\alpha+1}{\cosh\alpha-1} - \frac{1}{\omega}\right.$$
$$\left. - \frac{\omega}{12\sinh^2\alpha} + \frac{\omega^3}{720}\cdot\frac{13\cosh^2\alpha+5}{\sinh^4\alpha} - \dots\right\}. \qquad (24)$$

This formula will be found suitable for computing, except when $c/(b-a)$ is nearly equal to unity. In this case we get, by (14), (15), and (16),

$$\frac{2\mathrm{F}}{(v_1-v_2)^2} = \frac{r}{\omega}\left[\frac{1}{2x} + \psi'\left(\frac{\alpha}{\omega}\right)\cdot\frac{b+a}{3(b-a)^2}\cdot\left\{1+\frac{25ab-2(a^2+b^2)}{30ab}\,\frac{x}{b-a}\right\}\right.$$
$$\left. -\frac{1}{3}(k+\tfrac{1}{4})\frac{1}{b-a} - \frac{1}{1800}(331+948k-624k^2)\frac{x}{(b-a)^2}\right]$$
$$-\frac{\mathrm{C}\omega}{r}\cdot\frac{ab}{(b-a)^2}\cdot\left\{k+\frac{3+9k-2k^2}{10}\cdot\frac{2x}{b-a}\right\}. \qquad (25)$$

C can be computed by the formulæ given earlier in the paper and $\psi'(x)$[*] is found from

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{\mathrm{B}_1}{x^3} - \frac{\mathrm{B}_3}{x^5} + \frac{\mathrm{B}_5}{x^7} - \dots. \qquad (\mathrm{I})$$

If this converge too slowly we can write

$$\psi'(x) = \psi'(x+n) + \sum_0^{n-1}\frac{1}{(x+n)^2}, \qquad (\mathrm{J})$$

and use (I) to find $\psi'(x+n)$.

Finally, when x is very small, we have

$$\mathrm{F} = ab\,.\,(v_1-v_2)^2/4(b-a)x. \qquad (26)$$

In the case when the spheres are external to one another and the distance between them is microscopic, we previously found (*loc. cit. ante*) that

$$\mathrm{F} = ab\,.\,(v_1-v_2)^2/4(b+a)x.$$

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[*] A short table of the values of $\psi'(x)$ is given in 'Funktionentafeln,' Jahnke and Emde, p. 31.

When arranged as a spherical condenser, therefore, the attractive force is greater in the ratio of $b+a$ to $b-a$ when the distance apart is very minute.

When the electrostatic charge q on the inner condenser remains constant, the attractive force F′ is given by

$$F' = q^2/C^2 \,.\, \partial C/\partial c, \tag{27}$$

and formulæ for computation are obtained in exactly the same way as for the preceding problem.

In formula (26) if v_1-v_2 is given in electrostatic units, and a and b are in centimetres, then F is given in dynes. If v_1-v_2 is given in volts, we divide it by 300 to reduce it to electrostatic units. Similarly, if q in (27) be given in coulombs, we must multiply its value by 3×10^9.

3. *The Maximum Value of the Electric Stress.*

In determining the voltage at which the disruptive discharge inside the cavity takes place, a knowledge of the maximum value $R_{\text{max.}}$ of the electric stress on the dielectric is essential. In the case of spherical electrodes external to one another, the author* has shown how the value of this stress can be computed. In a similar way, it can readily be shown by the method of images that

$$R_{\text{max.}} = \frac{v_1-v_2}{a} \cdot \frac{\cosh^2\frac{1}{2}\alpha}{\sinh\frac{1}{2}\alpha}\left\{\frac{\sinh\frac{1}{2}\alpha}{\cosh^2\frac{1}{2}\alpha}+\frac{\sinh(\frac{1}{2}\alpha+\omega)}{\cosh^2(\frac{1}{2}\alpha+\omega)} + \frac{\sinh(\frac{1}{2}\alpha+2\omega)}{\cosh^2(\frac{1}{2}\alpha+2\omega)}+\dots\right\}. \tag{28}$$

When $c=0$, α is infinite and ω is $\log(b/a)$. In this case, therefore

$$R_{\text{max.}} = \frac{v_1-v_2}{a}\left\{1+\frac{1}{e^\omega}+\frac{1}{e^{2\omega}}+\dots\right\} = \frac{b}{a}\cdot\frac{v_1-v_2}{b-a}.$$

Expanding the third and subsequent terms of (28) in series, ascending by powers of $e^{-\alpha}$ and $e^{-\omega}$, and then summing the terms of the resulting series in a different order, we can show that

$$R_{\text{max.}} = \frac{v_1-v_2}{a}\left[1+\frac{ab\{b^2-c(c-a)\}}{\{b^2-c(c+a)\}^2}+e^{-\omega}\frac{(1+e^{-\alpha})^2}{1-e^{-\alpha}} \times\left\{\frac{e^{-\omega}}{1-e^{-\pi}}-3\frac{e^{-3\omega}}{1-e^{-3\omega}}e^{-(\alpha+2\omega)}+5\frac{e^{-5\omega}}{1-e^{-5\omega}}e^{-2(\alpha+2\omega)}-\dots\right\}\right]. \tag{29}$$

Except when $e^{-\omega}$ is nearly unity, this series is most suitable for computation. As an example, let us suppose that $b = 5$, $a = 2$, and $c = 2$. We find that $e^{-\omega} = 1/2$ and $e^{-\alpha} = 1/4$, and so, substituting in (29), we get

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* 'Proc. Phys. Soc. Lond.,' vol. 24, p. 22.

$$\mathrm{R}_{\text{max.}} = \tfrac{1}{2}(v_1-v_2)\{1+0{\cdot}8650520+\tfrac{25}{24}(1-0{\cdot}0267857+0{\cdot}0006301 -0{\cdot}0000135+0{\cdot}0000003-\ldots)\}$$
$$= 1{\cdot}4397297\,(v_1-v_2)/x,$$

where x = the distance between the spheres = 1.

When ω is small we can use the following formula deduced from (F) and (28).

$$\mathrm{R}_{\text{max.}} = \frac{v_1-v_2}{a}\left[\frac{1}{\omega}\cdot\frac{1+\cosh\alpha}{\sinh\alpha}+\frac{1}{2}-\frac{\omega}{6}\cdot\frac{1-\sinh^2\frac{1}{2}\alpha}{\sinh\alpha}\right.$$
$$-\frac{\omega^3}{720}\cdot\frac{1}{\sinh\frac{1}{2}\alpha\cosh^3\frac{1}{2}\alpha}\{\sinh^4\tfrac{1}{2}\alpha-18\sinh^2\tfrac{1}{2}\alpha+5\}$$
$$-\frac{\omega^5}{30240}\cdot\frac{1}{\sinh\frac{1}{2}\alpha\cosh^5\frac{1}{2}\alpha}$$
$$\left.\times\{179\sinh^4\tfrac{1}{2}\alpha+61-\sinh^6\tfrac{1}{2}\alpha-479\sinh^2\tfrac{1}{2}\alpha\}\right]. \qquad (30)$$

For example, when $b = 5$, $a = 2$, and $c = 2$, so that x is 1, we have

$$\mathrm{R}_{\text{max.}} = \tfrac{1}{2}(v_1-v_2)\{1{\cdot}202246+0{\cdot}25-0{\cdot}013478+0{\cdot}000759+0{\cdot}000176\}$$
$$= 1{\cdot}43970\,(v_1-v_2)/x.$$

In this case, therefore, where ω is nearly equal to 0·7, the error is about 1 in 48,000.

If in (30) we write $b-a-x$ for c, and neglect cubes and higher powers of $x/(b-a)$, we get, after lengthy algebraical work, that

$$\mathrm{R}_{\text{max.}} = \{1+(2b+a)\,x/3ab+[4\,(a+b)^2-ab]\,x^2/45a^2b^2+\ldots\}(v_1-v_2)/x. \quad (31)$$

As an example let $b = 5$, $a = 2$, and $x = 1$. Substituting in (31) we get

$$\mathrm{R}_{\text{max.}} = 1{\cdot}4413\,(v_1-v_2)/x.$$

Hence, although $x/(b-a)$ is as great as 1/3, the error is only about 1 in 700. If we write $-b$ for b in (31), we get the formula applicable to the case of external spheres. For external spheres, therefore,

$$\mathrm{R}_{\text{max.}} = \{1+(2b-a)\,x/3ab+[4\,(a-b)^2-ab]\,x^2/45a^2b^2\}\,(v_1-v_2)/x. \quad (32)$$

Comparing (31) and (32), we see that, for a given value of x and of v_1-v_2, the maximum electric stress is greater for the spherical condenser.

If we put $b = a$ in (34) we get

$$\mathrm{R}_{\text{max.}} = \{1+x/3a+x^2/45a^2+\ldots\}\,(v_1-v_2)/x,$$

a useful formula, which is applicable to the case of equal spherical electrodes when they are close together. This formula was first given by Schuster.*

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* 'Phil. Mag.,' vol. 29, p. 192 (1890).